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CONCENTRATION FIELDS OF RADIOACTIVE SUBSTANCES IN THE CASE OF UNDERGROUND BURIAL

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Concentration fields appearing in injection of solutions of radioactive substances into deep-lying strata have been calculated in the zero and first approximations on the basis of solutions constructed by asymptotic methods according to the concept of "exact, on the average," solutions. The residual term of the expansion has been evaluated; the conditions under which the first approximation is highly accurate have been found.

Introduction. Because of great environmental hazard, burial of solutions of radioactive isotopes in deep-lying underground horizons calls for reliable methods of predicting the dynamics of propagation of radioactive impurities, which makes it necessary to investigate convective diffusion in porous media [1, 2]. The problem on propagation of radioactive impurities in a deep-lying porous stratum into which a liquid with dissolved radioactive substances is injected is fundamental for underground burial of radioactive waste (and the waste of chemical industries) and for study of the consequences of underground nuclear explosions and the possibility of extracting gas by the method of injection of the isotopes under a gas-hydrate deposit.

Mathematical Formulation of the Mass-Transfer Problem. Figure 1 gives the geometry of the problem in a cylindrical coordinate system whose z_d axis is coincident with the axis of the well. The medium is presented by three regions with plane boundaries $z_d = \pm h$. The overlying and underlying strata are assumed to be impermeable; the central region of thickness $2h$ is porous. All the strata are taken to be homogeneous and anisotropic in diffusion properties.

Water with a radioactive pollutant is injected, via a well of small (compared to the distance to the point of observation) radius r_0, into an infinite horizontal stratum $-h < z_d < h$. The density of the impurity ρ_0 in the entering liquid is maintained constant for $r \le r_0$. The concentration of the pollutant in the stratum changes due to convective transfer along the direction r, diffusion along r and z, and the sources. Radioactive decay of the pollutant is considered as the sources.

Mathematical formulation of the mass-transfer problem for all the regions involves the diffusion equation with allowance for radioactive decay in the overlying

$$\frac{\partial \rho_{d1}}{\partial \tau} - D_{1z}\frac{\partial^2 \rho_{d1}}{\partial z_d^2} - D_{1r}\frac{1}{r_d}\frac{\partial}{\partial r_d}\left(r_d \frac{\partial \rho_{d1}}{\partial r_d}\right) = -\alpha\rho_{d1}\,, \quad \tau > 0\,, \quad r_d > 0\,, \quad z_d > h \tag{1}$$

and underlying

$$\frac{\partial \rho_{d2}}{\partial \tau} - D_{2z}\frac{\partial^2 \rho_{d2}}{\partial z_d^2} - D_{2r}\frac{1}{r_d}\frac{\partial}{\partial r_d}\left(r_d \frac{\partial \rho_{d2}}{\partial r_d}\right) = -\alpha\rho_{d2}\,, \quad \tau > 0\,, \quad r_d > 0\,, \quad z_d < -h \tag{2}$$

strata and the convective-diffusion equation with allowance for radioactive decay in the porous stratum

$$\frac{\partial \rho_d}{\partial \tau} - D_z\frac{\partial^2 \rho_d}{\partial z_d^2} - D_r\frac{1}{r_d}\frac{\partial}{\partial r_d}\left(r_d \frac{\partial \rho_d}{\partial r_d}\right) + \frac{v_0 r_0}{r_d}\frac{\partial \rho_d}{\partial r_d} = -\alpha\rho_d\,, \quad \tau > 0\,, \quad r_d > 0\,, \quad |z_d| < h\,. \tag{3}$$

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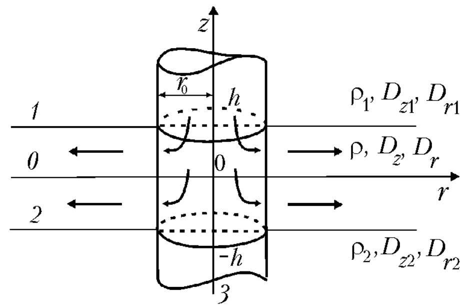

Fig. 1. Geometry of the problem: 0, 1, and 2) porous, overlying, and underlying strata respectively; 3) well.

The conjugation conditions represent the equalities of the densities and fluxes of a dissolved substance at the boundaries of the strata

$$\rho_{\rm d}\big|_{z_{\rm d}=h} = \rho_{\rm d1}\big|_{z_{\rm d}=h}, \quad \rho_{\rm d}\big|_{z_{\rm d}=-h} = \rho_{\rm d2}\big|_{z_{\rm d}=-h}, \tag{4}$$

$$D_z \frac{\partial \rho_{\rm d}}{\partial z_{\rm d}}\bigg|_{z_{\rm d}=h} = D_{1z} \frac{\partial \rho_{\rm d1}}{\partial z_{\rm d}}\bigg|_{z_{\rm d}=h}, \quad D_z \frac{\partial \rho_{\rm d}}{\partial z_{\rm d}}\bigg|_{z_{\rm d}=-h} = D_{2z} \frac{\partial \rho_{\rm d2}}{\partial z_{\rm d}}\bigg|_{z_{\rm d}=-h}. \tag{5}$$

The density of the pollutant in the well whose radius is assumed to be infinitesimal compared to the distance to the point of observation is

$$\rho_{\rm d}\big|_{r_{\rm d}=0} = \rho_0 . \tag{6}$$

We assume that at the initial instant of time, the pollutant density is

$$\rho_{\rm d}\big|_{\tau=0} = 0, \quad \rho_{\rm d1}\big|_{\tau=0} = 0, \quad \rho_{\rm d2}\big|_{\tau=0} = 0 . \tag{7}$$

Furthermore, the regularity conditions

$$\rho_{\rm d}\big|_{r_{\rm d}\to+\infty} = 0, \quad \rho_{\rm d1}\big|_{r_{\rm d}+z_{\rm d}\to+\infty} = 0, \quad \rho_{\rm d2}\big|_{r_{\rm d}+|z_{\rm d}|\to+\infty} = 0 . \tag{8}$$

are observed at infinity.

Let us pass to dimensionless coordinates

$$\rho = \frac{\rho_{\rm d}}{\rho_0}, \quad r = \frac{r_{\rm d}}{h}, \quad z = \frac{z_{\rm d}}{h}, \quad t = \frac{D_{1z}\tau}{h^2}, \quad {\rm Ad} = \frac{\alpha h^2}{D_{1z}}, \quad {}^2_1D = \frac{D_{2z}}{D_{1z}}, \quad {}^0_1D = \frac{D_z}{D_{1z}}, \quad {}^0_2D = \frac{D_z}{D_{2z}},$$

where the superscript and the subscript ahead of D are coincident with those of the diffusion coefficient in the numerator and the denominator of their ratio respectively (see Fig. 1). Also, we introduce an analog of the Péclet parameter Pe = $v_0 r_0/D_{1z}$. For injection volumes of $\sim 10^2$ m^3/day, the Pe number turns out to be of the order of 10^5 and the contribution of convective transfer begins to dominate that of radial diffusion. This enables us to disregard the term describing radial diffusion in Eq. (3). For the sake of simplicity we will also disregard the corresponding terms in Eqs. (1) and (2).

Expansion of the Mass-Transfer Problem in Asymptotic Parameter. Let us consider a more general problem obtained by introducing an arbitrary asymptotic parameter ε through formal replacement of the diffusion coefficient D_z by D_z/ε:

$$\frac{\partial \rho_1}{\partial t} - \frac{\partial^2 \rho_1}{\partial z^2} = -{\rm Ad}\, \rho_1, \quad t > 0, \quad r > 0, \quad z > 1, \tag{9}$$

$$\varepsilon \frac{\partial \rho}{\partial t} - {}^{0}_{1}D \frac{\partial^2 \rho}{\partial z^2} + \varepsilon \frac{\text{Pe}}{r} \frac{\partial \rho}{\partial r} = - \varepsilon \, \text{Ad} \, \rho \,, \quad t > 0 \,, \quad r > 0 \,, \quad |z| < 1 \,, \tag{10}$$

$$\frac{\partial \rho_2}{\partial t} - {}^{2}_{1}D \frac{\partial^2 \rho_2}{\partial z^2} = - \text{Ad} \, \rho_2 \,, \quad t > 0 \,, \quad r > 0 \,, \quad z < -1 \,, \tag{11}$$

with boundary conditions

$$\left. \frac{\partial \rho}{\partial z} \right|_{z=1} = {}^{1}_{0}D\varepsilon \left. \frac{\partial \rho_1}{\partial z} \right|_{z=1} \,, \quad \left. \frac{\partial \rho}{\partial z} \right|_{z=-1} = {}^{2}_{0}D\varepsilon \left. \frac{\partial \rho_2}{\partial z} \right|_{z=-1} \,, \tag{12}$$

$$\rho \, |_{z=1} = \rho_1 \, |_{z=1} \,, \quad \rho \, |_{z=-1} = \rho_2 \, |_{z=-1} \,, \tag{13}$$

$$\rho \, |_{r=0} = 1 \,, \quad \rho \, |_{t=0} = 0 \,, \quad \rho_1 \, |_{t=0} = 0 \,, \quad \rho_2 \, |_{t=0} = 0 \,, \tag{14}$$

$$\rho \, |_{r \to +\infty} = 0 \,, \quad \rho_1 \, |_{r+z \to +\infty} = 0 \,, \quad \rho_2 \, |_{r+|z| \to +\infty} = 0 \,. \tag{15}$$

We will seek the solution of problem (9)–(15), representing the density function ρ in each region by an asymptotic formula in parameter ε [3]:

$$\rho = \rho^{(0)} + \varepsilon \rho^{(1)} + \ldots + \varepsilon^n \rho^{(n)} + \theta^{(n)} \,, \quad \rho_1 = \rho_1^{(0)} + \varepsilon \rho_1^{(1)} + \ldots + \varepsilon^n \rho_1^{(n)} + \theta_1^{(n)} \,,$$

$$\rho_2 = \rho_2^{(0)} + \varepsilon \rho_2^{(1)} + \ldots + \varepsilon^n \rho_2^{(n)} + \theta_2^{(n)} \,. \tag{16}$$

Substituting expressions (16) into (9)–(15) and grouping the terms by the powers of the expansion parameter ε, we obtain

$$\left(\frac{\partial \rho_1^{(0)}}{\partial t} - \frac{\partial^2 \rho_1^{(0)}}{\partial z^2} + \text{Ad} \, \rho_1^{(0)} \right) + \varepsilon \left(\frac{\partial \rho_1^{(1)}}{\partial t} - \frac{\partial^2 \rho_1^{(1)}}{\partial z^2} + \text{Ad} \, \rho_1^{(1)} \right) + \ldots = 0 \,, \quad t > 0 \,, \quad r > 0 \,, \quad z > 1 \,; \tag{17}$$

$$\left(\frac{\partial \rho_2^{(0)}}{\partial t} - {}^{2}_{1}D \frac{\partial^2 \rho_2^{(0)}}{\partial z^2} + \text{Ad} \, \rho_2^{(0)} \right) + \varepsilon \left(\frac{\partial \rho_2^{(1)}}{\partial t} - {}^{2}_{1}D \frac{\partial^2 \rho_2^{(1)}}{\partial z^2} + \text{Ad} \, \rho_2^{(1)} \right) + \ldots = 0 \,, \quad t > 0 \,, \quad r > 0 \,, \quad z < -1 \,; \tag{18}$$

$$- {}^{0}_{1}D \frac{\partial^2 \rho^{(0)}}{\partial z^2} + \varepsilon \left(\frac{\partial \rho^{(0)}}{\partial t} - {}^{0}_{1}D \frac{\partial^2 \rho^{(1)}}{\partial z^2} + \frac{\text{Pe}}{r} \frac{\partial \rho^{(0)}}{\partial r} + \text{Ad} \, \rho^{(0)} \right)$$

$$+ \varepsilon^2 \left(\frac{\partial \rho^{(1)}}{\partial t} - {}^{0}_{1}D \frac{\partial^2 \rho^{(2)}}{\partial z^2} + \frac{\text{Pe}}{r} \frac{\partial \rho^{(1)}}{\partial r} + \text{Ad} \, \rho^{(1)} \right) + \ldots = 0 \,, \quad t > 0 \,, \quad r > 0 \,, \quad |z| < 1 \,; \tag{19}$$

$$\left.\frac{\partial\rho^{(0)}}{\partial z}\right|_{z=1}+\varepsilon\left(\left.\frac{\partial\rho^{(1)}}{\partial z}\right|_{z=1}-{}_{1}^{0}D\left.\frac{\partial\rho_1^{(0)}}{\partial z}\right|_{z=1}\right)+\ldots=0\,,\quad\left.\frac{\partial\rho^{(0)}}{\partial z}\right|_{z=-1}+\varepsilon\left(\left.\frac{\partial\rho^{(1)}}{\partial z}\right|_{z=-1}-{}_{0}^{2}D\left.\frac{\partial\rho_2^{(0)}}{\partial z}\right|_{z=-1}\right)+\ldots=0\,; \tag{20}$$

$$\left.\left(\rho^{(0)}+\varepsilon\rho^{(1)}+\ldots\right)\right|_{z=1}=\left.\left(\rho_1^{(0)}+\varepsilon\rho_1^{(1)}+\ldots\right)\right|_{z=1},\quad\left.\left(\rho^{(0)}+\varepsilon\rho^{(1)}+\ldots\right)\right|_{z=-1}=\left.\left(\rho_2^{(0)}+\varepsilon\rho_2^{(1)}+\ldots\right)\right|_{z=-1}; \tag{21}$$

$$\left.\left(\rho^{(0)}+\varepsilon\rho^{(1)}+\ldots\right)\right|_{t=0}=0\,,\quad\left.\left(\rho_1^{(0)}+\varepsilon\rho_1^{(1)}+\ldots\right)\right|_{t=0}=0\,,\quad\left.\left(\rho_2^{(0)}+\varepsilon\rho_2^{(1)}+\ldots\right)\right|_{t=0}=0\,,$$

$$\left.\left(\rho^{(0)}+\varepsilon\rho^{(1)}+\ldots\right)\right|_{r=0}=1\,; \tag{22}$$

$$\left.\left(\rho^{(0)}+\varepsilon\rho^{(1)}+\ldots\right)\right|_{r\to+\infty}=0\,,\quad\left.\left(\rho_1^{(0)}+\varepsilon\rho_1^{(1)}+\ldots\right)\right|_{r+z\to+\infty}=0\,,\quad\left.\left(\rho_2^{(0)}+\varepsilon\rho_2^{(1)}+\ldots\right)\right|_{r+|z|\to+\infty}=0\,. \tag{23}$$

An analysis of the formulation of the problem shows that the factors of the ε powers in (19) contain neighboring expansion coefficients and are linked in this sense. The procedure of unlinking has been carried out for solution of the corresponding equations.

Formulation of the Mass-Transfer Problem in the Zero Approximation. In the zero approximation, the pollutant density is only dependent on r and t $\rho^{(0)}=\rho^{(0)}(r,\ t)$ and is the solution of the following problem [2]:

$$\frac{\partial\rho_1^{(0)}}{\partial t}-\frac{\partial^2\rho_1^{(0)}}{\partial z^2}+\mathrm{Ad}\,\rho_1^{(0)}=0\,,\quad t>0\,,\quad r>0\,,\quad z>1\,; \tag{24}$$

$$\frac{\partial\rho_2^{(0)}}{\partial t}-{}_{1}^{2}D\,\frac{\partial^2\rho_2^{(0)}}{\partial z^2}+\mathrm{Ad}\,\rho_2^{(0)}=0\,,\quad t>0\,,\quad r>0\,,\quad z<-1\,; \tag{25}$$

$$\frac{\partial\rho^{(0)}}{\partial t}+\frac{\mathrm{Pe}}{r}\frac{\partial\rho^{(0)}}{\partial r}+\mathrm{Ad}\,\rho^{(0)}=\frac{1}{2}\left(\left.\frac{\partial\rho_1^{(0)}}{\partial z}\right|_{z=1}-{}_{1}^{2}D\left.\frac{\partial\rho_2^{(0)}}{\partial z}\right|_{z=-1}\right),\quad t>0\,,\quad r>0\,,\quad |z|<1\,; \tag{26}$$

$$\rho^{(0)}\big|_{z=1}=\rho_1^{(0)}\big|_{z=1}=\rho_2^{(0)}\big|_{z=-1}\,; \tag{27}$$

$$\rho^{(0)}\big|_{r=0}=1\,,\quad\rho^{(0)}\big|_{t=0}=0\,,\quad\rho_1^{(0)}\big|_{t=0}=0\,,\quad\rho_2^{(0)}\big|_{t=0}=0\,; \tag{28}$$

$$\rho^{(0)}\big|_{r\to+\infty}=0\,,\quad\rho_1^{(0)}\big|_{r+z\to+\infty}=0\,,\quad\rho_2^{(0)}\big|_{r+|z|\to+\infty}=0\,. \tag{29}$$

It is noteworthy that, unlike the initial problem which is the conjugation problem for parabolic equations, this problem is mixed, since the equation for the stratum is not parabolic. Furthermore, this equation contains the traces of the derivatives from the external regions.

Zero Approximation as the Solution of the Averaged Problem. Averaging (9)–(15) over z within the limits of the bearing stratum according to $\langle\rho\rangle = \frac{1}{2}\int\limits_{-1}^{1}\rho dz$, we obtain the formulation of the averaged parameterized problem which is coincident with (24)–(29) for the zero approximation of the pollutant density. Therefore, by virtue of the uniqueness of the solution, we have $\langle\rho\rangle = \rho^{(0)}$, i.e., the zero approximation describes the solutions of the initial problem that have been averaged in a certain manner.

If we average the initial nonparameterized problem, it is also coincident with the problem for the zero approximation of the density field of the pollutant. Coincidence of the averaged initial and parameterized problems makes the parameterization used in the present work significantly different from arbitrary parameterization which most frequently leads to the dependence of the averaged values on the parameter of asymptotic expansion.

Solution of the Problem in the Zero Approximation. In the Laplace–Carson transform space [4], problem (24)–(29) is reduced to ordinary differential equations whose solutions in the inverse-transform space are represented as

$$\rho^{(0)} = \frac{1}{2}\exp\left(-\frac{\mathrm{Ad}}{2\mathrm{Pe}}r^2\right)\Phi\left(t-\frac{r^2}{2\mathrm{Pe}}\right)$$

$$\times\left[\exp\left(-\frac{\sqrt{\mathrm{Ad}}\left(1+\sqrt{{}^{2}_{l}D}\right)}{4\mathrm{Pe}}r^2\right)\mathrm{erfc}\left(\frac{\left(1+\sqrt{{}^{2}_{l}D}\right)r^2}{4\sqrt{2\mathrm{Pe}}\sqrt{2\mathrm{Pe}\,t-r^2}}-\sqrt{\mathrm{Ad}\left(t-\frac{r^2}{2\mathrm{Pe}}\right)}\right)\right.$$

$$\left.+\exp\left(\frac{\sqrt{\mathrm{Ad}}\left(1+\sqrt{{}^{2}_{l}D}\right)}{4\mathrm{Pe}}r^2\right)\mathrm{erfc}\left(\frac{\left(1+\sqrt{{}^{2}_{l}D}\right)r^2}{4\sqrt{2\mathrm{Pe}}\sqrt{2\mathrm{Pe}\,t-r^2}}+\sqrt{\mathrm{Ad}\left(t-\frac{r^2}{2\mathrm{Pe}}\right)}\right)\right],$$

$$t>0\,,\quad r>0\,,\quad |z|<1\,; \tag{30}$$

$$\rho_1^{(0)} = \frac{1}{2}\exp\left(-\frac{\mathrm{Ad}}{2\mathrm{Pe}}r^2\right)\Phi\left(t-\frac{r^2}{2\mathrm{Pe}}\right)\left\{\exp\left(-\frac{\sqrt{\mathrm{Ad}}\left(1+\sqrt{{}^{2}_{l}D}\right)}{4\mathrm{Pe}}r^2-\sqrt{\mathrm{Ad}}\,(z-1)\right)\right.$$

$$\times\,\mathrm{erfc}\left(\frac{\left(1+\sqrt{{}^{2}_{l}D}\right)r^2+4\mathrm{Pe}\,(z-1)}{4\sqrt{2\mathrm{Pe}}\sqrt{2\mathrm{Pe}\,t-r^2}}-\sqrt{\mathrm{Ad}\left(t-\frac{r^2}{2\mathrm{Pe}}\right)}\right)+\exp\left(\frac{\sqrt{\mathrm{Ad}}\left(1+\sqrt{{}^{2}_{l}D}\right)}{4\mathrm{Pe}}r^2+\sqrt{\mathrm{Ad}}\,(z-1)\right)$$

$$\left.\times\,\mathrm{erfc}\left(\frac{\left(1+\sqrt{{}^{2}_{l}D}\right)r^2+4\mathrm{Pe}\,(z-1)}{4\sqrt{2\mathrm{Pe}}\sqrt{2\mathrm{Pe}\,t-r^2}}+\sqrt{\mathrm{Ad}\left(t-\frac{r^2}{2\mathrm{Pe}}\right)}\right)\right\},\quad t>0\,,\quad r>0\,,\quad z>1\,;$$

$$\rho_2^{(0)} = \frac{1}{2} \exp\left(-\frac{\text{Ad}}{2\text{Pe}} r^2\right) \Phi\left(t - \frac{r^2}{2\text{Pe}}\right) \left\{ \exp\left(-\frac{\sqrt{\text{Ad}}\left(1 + \sqrt{{}_1^2D}\right)}{4\text{Pe}} r^2 + \sqrt{\text{Ad}\, {}_2^1D}\,(z+1)\right)\right.$$

$$\times \operatorname{erfc}\left(\frac{\left(1 + \sqrt{{}_1^2D}\right) r^2 - 4\text{Pe}\sqrt{{}_2^1D}\,(z+1)}{4\sqrt{2\text{Pe}}\sqrt{2\text{Pe}\, t - r^2}} - \sqrt{\text{Ad}\left(t - \frac{r^2}{2\text{Pe}}\right)}\right) + \exp\left(\frac{\sqrt{\text{Ad}}\left(1 + \sqrt{{}_1^2D}\right)}{4\text{Pe}} r^2 - \sqrt{\text{Ad}\, {}_2^1D}\,(z+1)\right)$$

$$\left.\times \operatorname{erfc}\left(\frac{\left(1 + \sqrt{{}_1^2D}\right) r^2 + 4\text{Pe}\sqrt{{}_2^1D}\,(z+1)}{4\sqrt{2\text{Pe}}\sqrt{2\text{Pe}\, t - r^2}} + \sqrt{\text{Ad}\left(t - \frac{r^2}{2\text{Pe}}\right)}\right)\right\}, \quad t > 0, \quad r > 0, \quad z > 1 .$$

Mathematical Formulation of the Problem in the First Approximation. Equations (17)–(19) for the coefficients of ε take the form

$$\frac{\partial \rho_1^{(1)}}{\partial t} - \frac{\partial^2 \rho_1^{(1)}}{\partial z^2} + \text{Ad}\, \rho_1^{(1)} = 0 , \quad t > 0 , \quad r > 0 , \quad z > 1 , \tag{31}$$

$$\frac{\partial \rho_2^{(1)}}{\partial t} - {}_1^2D \frac{\partial^2 \rho_2^{(1)}}{\partial z^2} + \text{Ad}\, \rho_2^{(1)} = 0 , \quad t > 0 , \quad r > 0 , \quad z < -1 , \tag{32}$$

$$\frac{\partial \rho^{(1)}}{\partial t} + \frac{\text{Pe}}{r} \frac{\partial \rho^{(1)}}{\partial r} - {}_1^0D \frac{\partial^2 \rho^{(2)}}{\partial z^2} + \text{Ad}\, \rho^{(1)} = 0 , \quad t > 0 , \quad r > 0 , \quad |z| < 1 , \tag{33}$$

and the corresponding boundary and initial conditions are represented as

$$\left.\frac{\partial \rho^{(1)}}{\partial z}\right|_{z=1} - {}_0^1D \left.\frac{\partial \rho_1^{(0)}}{\partial z}\right|_{z=1} = 0 , \quad \left.\frac{\partial \rho^{(1)}}{\partial z}\right|_{z=-1} - {}_0^2D \left.\frac{\partial \rho_2^{(0)}}{\partial z}\right|_{z=-1} = 0 ; \tag{34}$$

$$\rho^{(1)}\big|_{z=1} = \rho_1^{(1)}\big|_{z=1} , \quad \rho^{(1)}\big|_{z=-1} = \rho_2^{(1)}\big|_{z=-1} , \quad \rho^{(1)}\big|_{t=0} = \rho_1^{(1)}\big|_{t=0} = \rho_2^{(1)}\big|_{t=0} = 0 ;$$

$$\rho^{(1)}\big|_{r\to+\infty} = 0 , \quad \rho_1^{(1)}\big|_{r+z\to+\infty} = 0 , \quad \rho_2^{(1)}\big|_{r+|z|\to+\infty} = 0 ; \tag{35}$$

$$\rho^{(1)}\big|_{r=0} = 0 . \tag{36}$$

The solution of the problem is sought in the form of a quadratic polynomial for z [2]

$$\rho^{(1)} = {}_0^1D \left(\frac{z^2}{2} E(r, t) + zF(r, t) + Q(r, t)\right). \tag{37}$$

For the sake of convenience we introduce the operator $\widehat{L} = \frac{\partial}{\partial t} + \frac{\text{Pe}\,\partial}{r\partial r} + \text{Ad}$. Allowing for (36) and for the linearity of the operator $\widehat{L}$, from (33) we obtain

$$\frac{\partial^2 \rho^{(2)}}{\partial z^2} = {}_0^1D^2 \left(\frac{z^2}{2} \widehat{L}E(r,t) + z\widehat{L}F(r,t) + \widehat{L}Q(r,t) \right). \tag{38}$$

The auxiliary functions $E(r, t)$ and $F(r, t)$ are determined in terms of the traces of the derivatives from the external regions

$$E(r,t) = \frac{1}{2} \left(\left. \frac{\partial \rho_1^{(0)}}{\partial z} \right|_{z=1} - {}_1^2D \left. \frac{\partial \rho_2^{(0)}}{\partial z} \right|_{z=-1} \right), \quad F(r,t) = \frac{1}{2} \left(\left. \frac{\partial \rho_1^{(0)}}{\partial z} \right|_{z=1} + {}_1^2D \left. \frac{\partial \rho_2^{(0)}}{\partial z} \right|_{z=-1} \right), \tag{39}$$

and the function $Q(r, t)$ is determined from the equation

$$\widehat{L}Q(r,t) = \frac{{}_1^0D}{2} \left(\left. \frac{\partial \rho_1^{(1)}}{\partial z} \right|_{z=1} - {}_1^2D \left. \frac{\partial \rho_2^{(1)}}{\partial z} \right|_{z=-1} \right) - \frac{1}{6} \widehat{L}E(r,t). \tag{40}$$

An equation for determining the first expansion coefficient is obtained through substitution of expressions (39) and (40) into (38)

$$\widehat{L}\rho^{(1)} = \frac{{}_0^1D}{4} \left(z^2 - \frac{1}{3} \right) \widehat{L} \left(\left. \frac{\partial \rho_1^{(0)}}{\partial z} \right|_{z=1} - {}_1^2D \left. \frac{\partial \rho_2^{(0)}}{\partial z} \right|_{z=-1} \right) - \frac{{}_0^1D}{2} z\widehat{L} \left(\left. \frac{\partial \rho_1^{(0)}}{\partial z} \right|_{z=1} + {}_1^2D \left. \frac{\partial \rho_2^{(0)}}{\partial z} \right|_{z=-1} \right)$$

$$+ \frac{1}{2} \left(\left. \frac{\partial \rho_1^{(1)}}{\partial z} \right|_{z=1} - {}_1^2D \left. \frac{\partial \rho_2^{(1)}}{\partial z} \right|_{z=-1} \right). \tag{41}$$

Also, the problem for determining the first expansion coefficient involves Eqs. (31) and (32) for the ambient medium.

The formulated problem (31), (32), (41), (34), and (35) with condition (36) has a trivial solution; therefore, condition (36) is in excess and must be attenuated. This is possible from the following considerations. The solution in the zero approximation describes the mean values and holds for large and small times. The first approximation is a correction to the zero one, in which we can use modified boundary conditions. However, according to the theory, these conditions can be restored using boundary-value functions. The region of high accuracy of calculations changes. However, to determine the "high-accuracy" region we must solve the problem for the residual term based on which the conclusion on the exactness of the first approximation is drawn.

Problem for the Residual Term. Substituting the expansion (16) for $n = 1$ into (9)–(15), we obtain the problem for the residual term the construction of whose exact solution presents substantial difficulties. We restrict ourselves to investigation of the problem averaged over the stratum thickness:

$$\frac{\partial \theta_1}{\partial t} - \frac{\partial^2 \theta_1}{\partial z^2} = -\,\text{Ad}\,\theta_1, \quad t > 0, \quad r > 0, \quad z > 1, \tag{42}$$

$$\frac{\partial \langle\theta\rangle}{\partial t} + \frac{\text{Pe}}{r}\frac{\partial \langle\theta\rangle}{\partial r} + \text{Ad}\,\langle\theta\rangle - \frac{1}{2}\left(\left.\frac{\partial\theta_1}{\partial z}\right|_{z=1} - {}^{2}_{1}D \left.\frac{\partial\theta_2}{\partial z}\right|_{z=-1}\right)$$

$$= -\varepsilon\left(\widehat{L}\,\langle\rho^{(1)}\rangle - \frac{1}{2}\left(\left.\frac{\partial\rho_1^{(1)}}{\partial z}\right|_{z=1} - {}^{2}_{1}D\left.\frac{\partial\rho_2^{(1)}}{\partial z}\right|_{z=-1}\right)\right), \quad t>0\,, \quad r>0\,, \quad |z|<1\,, \tag{43}$$

$$\frac{\partial\theta_2}{\partial t} - {}^{2}_{1}D\frac{\partial^2\theta_2}{\partial z^2} = -\,\text{Ad}\,\theta_2\,, \quad t>0\,, \quad r>0\,, \quad z<-1\,, \tag{44}$$

$$\langle\theta\rangle = \theta_1\,|_{z=1} = \theta_2\,|_{z=-1}\,, \tag{45}$$

$$\langle\theta\rangle\,|_{r=0} = -\,\varepsilon\,\langle\rho^{(1)}\rangle\,|_{r=0}\,, \tag{46}$$

$$\langle\theta\rangle\,|_{t=0} = 0\,, \quad \theta_1\,|_{t=0} = 0\,, \quad \theta_2\,|_{t=0} = 0\,, \tag{47}$$

$$\langle\theta\rangle\,|_{r\to+\infty} = 0\,, \quad \theta_1\,|_{r+z\to+\infty} = 0\,, \quad \theta_2\,|_{r+|z|\to+\infty} = 0\,. \tag{48}$$

It is easily established that problem (42)–(48) has a trivial solution if and only if

$$\langle\rho^{(1)}\rangle\,|_{r=0} = 0\,, \tag{49}$$

$$\widehat{L}\,\langle\rho^{(1)}\rangle - \frac{1}{2}\left(\left.\frac{\partial\rho_1^{(1)}}{\partial z}\right|_{z=1} - {}^{2}_{1}D\left.\frac{\partial\rho_2^{(1)}}{\partial z}\right|_{z=-1}\right) = 0\,, \tag{50}$$

i.e., if there are no sources in the averaged problem for the residual term. Averaging (33) with account for (34), we may show that (50) is observed identically. Consequently, for the averaged problem for the residual term to have a trivial solution, it is sufficient to observe condition (49).

The asymptotic expansion constructed in such a manner possesses an important property; it implies that the mean value of the residual term vanishes. Naturally, this makes the solution much more valuable for practical applications. By virtue of this fact, it is expedient to single out the class of solutions in question in asymptotic solutions. The asymptotic approximation of problem (1)–(8), constructed on condition that the solution of the averaged problem for the residual term is trivial will be called *the exact, on the average*, *asymptotic solution*.

Thus, we have obtained, from (46), the additional nonlocal boundary condition (49) for the exact, on the average, solution.

Solution of the Problem for the First Expansion Coefficient. Problem (31)–(35) with condition (49) is solved by the method of integral Laplace–Carson transformations, and the solution has the following form respectively for the porous, overlying, and underlying strata:

$$\rho^{(1)} = {}_0^1D \, \frac{\sqrt{\mathrm{Ad}}}{4} \exp\left(-\frac{\mathrm{Ad}}{2\mathrm{Pe}} r^2\right) \Phi\left(t - \frac{r^2}{2\mathrm{Pe}}\right)$$

$$\times \Bigg[\frac{2\sqrt{2\mathrm{Pe}}\left(1+\sqrt{{}_1^2D}\right)}{\sqrt{\pi}\,\sqrt{\mathrm{Ad}\,(2\mathrm{Pe}\,t - r^2)}} \Bigg(\frac{1-\sqrt{{}_1^2D}+{}_1^2D}{12\mathrm{Pe}\,(2\mathrm{Pe}\,t - r^2)} r^4 + \left(\frac{1}{6} - \frac{z^2}{2} - z\,\frac{1-\sqrt{{}_1^2D}}{1+\sqrt{{}_1^2D}}\right)\Bigg)$$

$$\times \exp\left(-\frac{\left(1+\sqrt{{}_1^2D}\right)^2}{32\mathrm{Pe}\,(2\mathrm{Pe}\,t - r^2)} r^4 - \mathrm{Ad}\left(t - \frac{r^2}{2\mathrm{Pe}}\right)\right)$$

$$+ \left(\sqrt{\mathrm{Ad}}\,\frac{1-\sqrt{{}_1^2D}+{}_1^2D}{3\mathrm{Pe}} r^2 - \left(1+\sqrt{{}_1^2D}\right)\left(\frac{1}{6} - \frac{z^2}{2} - z\,\frac{1-\sqrt{{}_1^2D}}{1+\sqrt{{}_1^2D}}\right)\right)$$

$$\times \exp\left(\frac{\sqrt{\mathrm{Ad}}\left(1+\sqrt{{}_1^2D}\right)}{4\mathrm{Pe}} r^2\right) \mathrm{erfc}\left(\frac{1+\sqrt{{}_1^2D}}{4\sqrt{2\mathrm{Pe}}\,\sqrt{2\mathrm{Pe}\,t - r^2}} r^2 + \sqrt{\mathrm{Ad}\left(t - \frac{r^2}{2\mathrm{Pe}}\right)}\right)$$

$$+ \left(\sqrt{\mathrm{Ad}}\,\frac{1-\sqrt{{}_1^2D}+{}_1^2D}{3\mathrm{Pe}} r^2 + \left(1+\sqrt{{}_1^2D}\right)\left(\frac{1}{6} - \frac{z^2}{2} - z\,\frac{1-\sqrt{{}_1^2D}}{1+\sqrt{{}_1^2D}}\right)\right)$$

$$\times \exp\left(-\frac{\sqrt{\mathrm{Ad}}\left(1+\sqrt{{}_1^2D}\right)}{4\mathrm{Pe}} r^2\right) \mathrm{erfc}\left(\frac{1+\sqrt{{}_1^2D}}{4\sqrt{2\mathrm{Pe}}\,\sqrt{2\mathrm{Pe}\,t - r^2}} r^2 - \sqrt{\mathrm{Ad}\left(t - \frac{r^2}{2\mathrm{Pe}}\right)}\right)\Bigg], \quad r > 0, \quad t > 0, \quad |z| < 1;$$

$$\rho_1^{(1)} = {}_0^1D \, \frac{\sqrt{\mathrm{Ad}}}{6} \exp\left(-\frac{\mathrm{Ad}}{2\mathrm{Pe}} r^2\right) \Phi\left(t - \frac{r^2}{2\mathrm{Pe}}\right) \tag{51}$$

$$\times \Bigg[\frac{2\sqrt{2\mathrm{Pe}}}{\sqrt{\pi}\,\sqrt{\mathrm{Ad}\,(2\mathrm{Pe}\,t - r^2)}} \left(\frac{1+{}_1^2D-\sqrt{{}_1^2D}}{2\,(2\mathrm{Pe}\,t - r^2)} r^2 \left(\frac{1+\sqrt{{}_1^2D}}{4\mathrm{Pe}} r^2 + z - 1\right) - 2 + \sqrt{{}_1^2D}\right)$$

$$\times \exp\left(-\frac{\left(\left(1+\sqrt{{}_1^2D}\right) r^2 + 4\mathrm{Pe}\,(z-1)\right)^2}{64\mathrm{Pe}^2\,(2\mathrm{Pe}\,t - r^2)} - \mathrm{Ad}\left(t - \frac{r^2}{2\mathrm{Pe}}\right)\right)$$

$$
+\left(\frac{\sqrt{\mathrm{Ad}}\left(1+{}_{1}^{2}D-\sqrt{{}_{1}^{2}D}\right)}{2\mathrm{Pe}}\,r^{2}-2+\sqrt{{}_{1}^{2}D}\right)\exp\left(-\frac{\sqrt{\mathrm{Ad}}\left(1+\sqrt{{}_{1}^{2}D}\right)}{4\mathrm{Pe}}\,r^{2}-\sqrt{\mathrm{Ad}}\,(z-1)\right)
$$

$$
\times\,\mathrm{erfc}\left(\frac{\left(1+\sqrt{{}_{1}^{2}D}\right)r^{2}+4\mathrm{Pe}\,(z-1)}{4\sqrt{2\mathrm{Pe}}\,\sqrt{2\mathrm{Pe}\,t-r^{2}}}-\sqrt{\mathrm{Ad}\left(t-\frac{r^{2}}{2\mathrm{Pe}}\right)}\right)
$$

$$
+\left(\frac{\sqrt{\mathrm{Ad}}\left(1+{}_{1}^{2}D-\sqrt{{}_{1}^{2}D}\right)}{2\mathrm{Pe}}\,r^{2}+2-\sqrt{{}_{1}^{2}D}\right)\exp\left(\frac{\sqrt{\mathrm{Ad}}\left(1+\sqrt{{}_{1}^{2}D}\right)}{4\mathrm{Pe}}\,r^{2}+\sqrt{\mathrm{Ad}}\,(z-1)\right)
$$

$$
\times\,\mathrm{erfc}\left(\frac{\left(1+\sqrt{{}_{1}^{2}D}\right)r^{2}+4\mathrm{Pe}\,(z-1)}{4\sqrt{2\mathrm{Pe}}\,\sqrt{2\mathrm{Pe}\,t-r^{2}}}+\sqrt{\mathrm{Ad}\left(t-\frac{r^{2}}{2\mathrm{Pe}}\right)}\right)\Bigg],\quad r>0\,,\quad t>0\,,\quad z>1\,;
$$

$$
\rho_{2}^{(1)}={}_{0}^{1}D\,\frac{\sqrt{\mathrm{Ad}}}{6}\exp\left(-\frac{\mathrm{Ad}}{2\mathrm{Pe}}\,r^{2}\right)\Phi\left(t-\frac{r^{2}}{2\mathrm{Pe}}\right)
$$

$$
\times\Bigg[\frac{2\sqrt{2\mathrm{Pe}}}{\sqrt{\pi}\,\sqrt{\mathrm{Ad}}\,(2\mathrm{Pe}\,t-r^{2})}\left(\frac{1+{}_{1}^{2}D-\sqrt{{}_{1}^{2}D}}{2\,(2\mathrm{Pe}\,t-r^{2})}\,r^{2}\left(\frac{1+\sqrt{{}_{1}^{2}D}}{4\mathrm{Pe}}\,r^{2}-\sqrt{{}_{2}^{1}D}\,(z+1)\right)+1-2\sqrt{{}_{1}^{2}D}\right)
$$

$$
\times\exp\left(-\frac{\left(\left(1+\sqrt{{}_{1}^{2}D}\right)r^{2}-4\mathrm{Pe}\sqrt{{}_{2}^{1}D}\,(z+1)\right)^{2}}{32\mathrm{Pe}\,(2\mathrm{Pe}\,t-r^{2})}-\mathrm{Ad}\left(t-\frac{r^{2}}{2\mathrm{Pe}}\right)\right)
$$

$$
+\left(\frac{\sqrt{\mathrm{Ad}}\left(1+{}_{1}^{2}D-\sqrt{{}_{1}^{2}D}\right)}{2\mathrm{Pe}}\,r^{2}+1-2\sqrt{{}_{1}^{2}D}\right)\exp\left(-\frac{\sqrt{\mathrm{Ad}}\left(1+\sqrt{{}_{1}^{2}D}\right)}{4\mathrm{Pe}}\,r^{2}+\sqrt{\mathrm{Ad}}\,{}_{2}^{1}D\,(z+1)\right)
$$

$$
\times\,\mathrm{erfc}\left(\frac{\left(1+\sqrt{{}_{1}^{2}D}\right)r^{2}-4\mathrm{Pe}\sqrt{{}_{2}^{1}D}\,(z+1)}{4\sqrt{2\mathrm{Pe}}\,\sqrt{2\mathrm{Pe}\,t-r^{2}}}-\sqrt{\mathrm{Ad}\left(t-\frac{r^{2}}{2\mathrm{Pe}}\right)}\right)
$$

$$
+\left(\frac{\sqrt{\mathrm{Ad}}\left(1+{}_{1}^{2}D-\sqrt{{}_{1}^{2}D}\right)}{2\mathrm{Pe}}\,r^{2}-1+2\sqrt{{}_{1}^{2}D}\right)\exp\left(\frac{\sqrt{\mathrm{Ad}}\left(1+\sqrt{{}_{1}^{2}D}\right)}{4\mathrm{Pe}}\,r^{2}-\sqrt{\mathrm{Ad}}\,{}_{2}^{1}D\,(z+1)\right)
$$

$$
\times\,\mathrm{erfc}\left(\frac{\left(1+\sqrt{{}_{1}^{2}D}\right)r^{2}-4\mathrm{Pe}\sqrt{{}_{2}^{1}D}\,(z+1)}{4\sqrt{2\mathrm{Pe}}\,\sqrt{2\mathrm{Pe}\,t-r^{2}}}+\sqrt{\mathrm{Ad}\left(t-\frac{r^{2}}{2\mathrm{Pe}}\right)}\right)\Bigg],\quad r>0\,,\quad t>0\,,\quad z<-1\,.
$$

It is noteworthy that for $r = 0$ the first expansion coefficient of (51) is dependent on the variable z and does not vanish, i.e., the constructed solution does not satisfy boundary condition (36). To eliminate the above drawback we must construct boundary-layer functions [5] in the vicinity of $r = 0$.

Construction of the Boundary-Layer Solution. We represent the solution of problem (9)–(15) as

$$\rho_1 = \hat{\rho}_1 + \Pi_1 \,, \quad \rho = \hat{\rho} + \Pi \,, \quad \rho_2 = \hat{\rho}_2 + \Pi_2 \,, \tag{52}$$

where $\hat{\rho} = \hat{\rho}(r, z, t)$ is the regular part and $\Pi = \Pi(y, z, t)$ is the boundary-layer part of the expansion of the solution in asymptotic parameter; $y = r^2/2\varepsilon$ is the extended variable. Substituting (52) into (9)–(15) and applying the Laplace–Carson transformation to the variable t, we obtain, in the transform space, the problem for boundary-layer functions

$$\Pi^{\text{tr}} = \Pi^{(0)\text{tr}} + \varepsilon\Pi^{(1)\text{tr}} + \ldots \,, \quad \Pi_1^{\text{tr}} = \Pi_1^{(0)\text{tr}} + \varepsilon\Pi_1^{(1)\text{tr}} + \ldots \,, \quad \Pi_2^{\text{tr}} = \Pi_2^{(0)\text{tr}} + \varepsilon\Pi_2^{(1)\text{tr}} + \ldots \,.$$

The problem for zero coefficients has only a trivial solution. The problem for first coefficients is broken into three independent problems. The first problem for the boundary-layer function in the stratum has the form

$$\frac{\partial^2 \Pi^{(1)\text{tr}}}{\partial z^2} - {}_0^1D \,\text{Pe}\, \frac{\partial \Pi^{(1)\text{tr}}}{\partial y} = 0 \,, \quad y > 0 \,, \quad |z| < 1 \,; \quad \left.\frac{\partial \Pi^{(1)\text{tr}}}{\partial z}\right|_{z=1} = \left.\frac{\partial \Pi^{(1)\text{tr}}}{\partial z}\right|_{z=-1} = 0 \,;$$

$$\Pi^{(1)\text{tr}}\big|_{y=0} = -\rho^{(1)\text{tr}}\big|_{y=0} = -\frac{{}_0^1D \sqrt{p + \text{Ad}}}{2} \left[\left(\frac{1}{6} - \frac{z^2}{2}\right)\left(1 + \sqrt{{}_1^2D}\right) - z\left(1 - \sqrt{{}_1^2D}\right)\right],$$

$$\Pi^{(1)\text{tr}}\big|_{y\to\infty} = 0 \,.$$

Its solution found by the method of separation of variables enables us to determine the boundary values of the first boundary-layer coefficients for the ambient half-spaces. Passing to inverse transforms, we obtain expressions for the first coefficient of expansion of the boundary-layer function in the stratum

$$\Pi^{(1)} = \frac{{}_0^1D}{\pi^2}\left(\frac{\exp(-t\,\text{Ad})}{\sqrt{\pi t}} + \sqrt{\text{Ad}}\,\text{erf}\,(\sqrt{t\,\text{Ad}})\right)\left\{4\left(1 - \sqrt{{}_1^2D}\right)\sin\left(\frac{\pi}{2}z\right)\exp\left(-\left(\frac{\pi}{2}\right)^2 \frac{{}_1^0D}{\text{Pe}}\,y\right)\right.$$
$$+ \sum_{n=1}^{\infty}\left(4\frac{(-1)^n}{(1+2n)^2}\left(1 - \sqrt{{}_1^2D}\right)\sin\left[\left(\frac{\pi}{2} + \pi n\right)z\right]\exp\left(-\left(\frac{\pi}{2} + \pi n\right)^2 \frac{{}_1^0D}{\text{Pe}}\,y\right)\right.$$
$$\left.\left.+ \frac{(-1)^n}{n^2}\left(1 + \sqrt{{}_1^2D}\right)\cos(\pi n z)\exp\left(-(\pi n)^2 \frac{{}_1^0D}{\text{Pe}}\,y\right)\right)\right\}, \tag{53}$$

in the overlying and underlying strata respectively

$$\Pi_1^{(1)} = \left\{\frac{1}{\sqrt{\pi t}}\exp\left(-t\,\text{Ad} - \frac{(z-1)^2}{4t}\right) - \frac{\sqrt{\text{Ad}}}{2}\left[\exp\left[(z-1)\sqrt{\text{Ad}}\right]\text{erfc}\left(\frac{z-1}{2\sqrt{t}} + \sqrt{t\,\text{Ad}}\right)\right.\right.$$
$$\left.\left.- \exp\left[-(z-1)\sqrt{\text{Ad}}\right]\text{erfc}\left(\frac{z-1}{2\sqrt{t}} - \sqrt{t\,\text{Ad}}\right)\right]\right\}\left(\frac{4\,{}_0^1D}{\pi^2}\left(1 - \sqrt{{}_1^2D}\right)\exp\left[-\left(\frac{\pi}{2}\right)^2\frac{{}_1^0D}{\text{Pe}}\,y\right]\right.$$
$$\left.+ \frac{{}_0^1D}{\pi^2}\sum_{n=1}^{\infty}\left(4\frac{(-1)^{2n}}{(1+2n)^2}\left(1 - \sqrt{{}_1^2D}\right)\exp\left[-\left(\frac{\pi}{2} + \pi n\right)^2\frac{{}_1^0D}{\text{Pe}}\,y\right] + \frac{(-1)^{2n}}{n^2}\left(1 + \sqrt{{}_1^2D}\right)\exp\left[-(\pi n)^2\frac{{}_1^0D}{\text{Pe}}\,y\right]\right)\right),$$

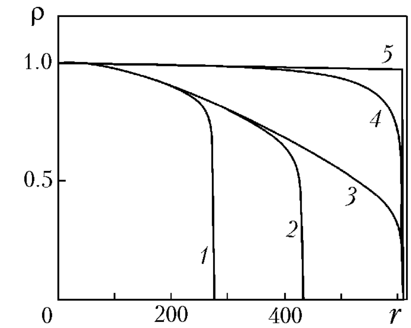

Fig. 2. Density of radioactive impurities in the first approximation vs. radial coordinate at the center of the stratum $z = 0$ for different times: 1) $t = 0.002$, 2) 0.005, and 3) 0.02 and Ad = 80, 4) 0.02 and Ad = 0, 5) 0.02 and Ad = 0, the diffusion-free approximation.

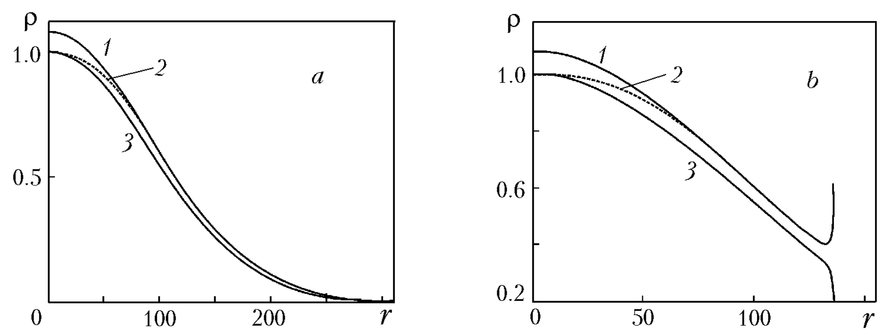

Fig. 3. Density of radioactive impurities vs. radial coordinate for the dimensionless time $t = 0.01$ (a) and $t = 0.001$ (b): 1 and 2) without and with allowance for the boundary-layer correction respectively; 3) zero approximation for $z = 0$.

$$\Pi_2^{(1)} = \left\{ \frac{1}{\sqrt{\pi t}} \exp\left(-t\,\mathrm{Ad} - \frac{{}^{2}_{1}D\,(z+1)^2}{4t} \right) - \frac{\sqrt{\mathrm{Ad}}}{2} \left[\exp\left[-(z+1)\sqrt{{}^{2}_{1}D\,\mathrm{Ad}} \right] \mathrm{erfc}\left(\sqrt{t\,\mathrm{Ad}} - \frac{(z+1)\sqrt{{}^{2}_{1}D}}{2\sqrt{t}} \right) \right.\right.$$

$$\left.\left. - \exp\left[-(z+1)\sqrt{{}^{2}_{1}D\,\mathrm{Ad}} \right] \mathrm{erfc}\left(-\frac{(z+1)\sqrt{{}^{2}_{1}D}}{2\sqrt{t}} - \sqrt{t\,\mathrm{Ad}} \right) \right] \right\} \left(\frac{4\,{}^{1}_{0}D}{\pi^2} \left(\sqrt{{}^{2}_{1}D} - 1 \right) \exp\left[-\left(\frac{\pi}{2} \right)^2 \frac{{}^{0}_{1}D}{\mathrm{Pe}} y \right] \right.$$

$$\left. + \frac{{}^{1}_{0}D}{\pi^2} \sum_{n=1}^{\infty} \left(4 \frac{(-1)^{2n+1}}{(1+2n)^2} \left(1 - \sqrt{{}^{2}_{1}D} \right) \exp\left[-\left(\frac{\pi}{2} + \pi n \right)^2 \frac{{}^{0}_{1}D}{\mathrm{Pe}} y \right] + \frac{(-1)^{2n}}{n^2} \left(1 + \sqrt{{}^{2}_{1}D} \right) \exp\left[-(\pi n)^2 \frac{{}^{0}_{1}D}{\mathrm{Pe}} y \right] \right) \right).$$

The solution of the nonstationary problem (1)–(8) in the asymptotic approximation with allowance for the boundary layer is represented by the sum of expressions (30), (51), and (53) in the stratum and by the corresponding expressions for the ambient media

$$\rho = \rho^{(0)} + \varepsilon\left(\rho^{(1)} + \Pi^{(1)} \right), \quad \rho_1 = \rho_1^{(0)} + \varepsilon\left(\rho_1^{(1)} + \Pi_1^{(1)} \right), \quad \rho_2 = \rho_2^{(0)} + \varepsilon\left(\rho_2^{(1)} + \Pi_2^{(1)} \right).$$

Analysis of the Results. Figure 2 gives the dependences (30) of the radioactive-pollutant density on the radial coordinate in the zero approximation for different injection times with allowance (curves 1, 2, 3) and without allowance (curve 4) for radioactive decay and in the diffusion-free approximation (curve 5). In the calculations, we have taken Pe = $2\cdot 10^7$, ${}^{1}_{0}D = 0.01$, ${}^{2}_{1}D = 1$, and Ad = 80, which corresponds to the half-life period $T_{1/2} = 28$ years. The figure enables us to evaluate the contribution of diffusion and radioactive decay to the change in the pollutant density.

It is seen that allowance for the radioactive decay (comparison of curves 3 and 4) is of higher priority compared to allowance for the mass exchange between the stratum (comparison of curves 4 and 5) and the surrounding rocks by diffusion. Also, it follows from the figure that the contribution of diffusion is substantial in the zone of the radioactive-contamination front where it is comparable to the density of the substance in the solution. The dimensions of the contamination zone are determined using the zero approximation.

Figure 3 shows that allowance for the boundary-layer solution eliminates the drawback of the first approximation, implying that the radioactive-impurity density is higher than unity (curve 1). In the calculations, we have taken $\mathrm{Pe} = 2\cdot 10^7$, ${}^{1}_{0}D = 0.01$, ${}^{2}_{1}D = 1$, $\varepsilon = 0.05$, and $\mathrm{Ad} = 2200$, which corresponds to the half-life period $T_{1/2} = 1$ year. The boundary-layer correction makes the curve's behavior correspond to the conditions of the problem (comparison of curves 1 and 2) and in so doing refines the first approximation. It is noteworthy that, for small times, allowance for the correction of the first expansion coefficient leads to a refinement of to 20% in the calculated parameter (comparison of curves 1 and 3 in Fig. 3b). At the same time, Fig. 3 illustrates the importance of allowing for the boundary-layer function in the near-wall zone ($r < 70$). As is seen in Fig. 3b, an abrupt change in the density occurs at the boundary of the front of pollutant propagation, which points to the presence of the internal moving boundary layer the regularities of whose formation have not been investigated and the mathematical techniques of eliminating have not been developed.

Conclusions. Based on the asymptotic method, we have obtained the approximate asymptotic solution of the problem on underground burial of waste with allowance for boundary-layer functions which enable us to calculate, with a high degree of accuracy, the fields of radioactive impurities in underground horizons at any distances from the injection well thus refining the prediction of radioactive pollutions to ensure environmental safety.

NOTATION

D_z, D_{1z}, D_{2z}, D_r, D_{1r}, and D_{2r}, diffusion coefficients in the vertical and radial directions respectively in the porous, overlying, and underlying strata, $\mathrm{m}^2/\mathrm{sec}$; E, F, and Q, auxiliary functions; h, stratum half-thickness, m; $\hat{L}$, operator; Pe, analog of the Péclet parameter; p, parameter of the Laplace–Carson transformation; r_0, well radius, m; r_d, z_d, r, and z, dimensional and dimensionless cylindrical coordinates, m; v_0, velocity of convective transfer of the pollutant at a distance r_0 from the well axis, m/sec; y, extended variable; α and Ad, dimensional and dimensionless constants of radioactive decay, 1/sec; ε, parameter of asymptotic expansion; θ, θ_1, and θ_2, dimensionless residual terms of expansions; $\Pi(y, z, t)$, $\Pi_1(y, z, t)$, and $\Pi_2(y, z, t)$, boundary-layer part of the expansion of the solution in asymptotic parameter for the porous, overlying, and underlying strata respectively; ρ_0, liquid density in the well, $\mathrm{kg/m}^3$; ρ_d, $\rho_{1\mathrm{d}}$, $\rho_{2\mathrm{d}}$, ρ, ρ_1, and ρ_2, dimensional and dimensionless densities of the pollutant in the porous, overlying, and underlying strata, $\mathrm{kg/m}^3$; τ and t, dimensional and dimensionless observation time, sec; $\Phi(t)$, Heaviside unit function; $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp(-y^2)dy$. Subscripts: tr, transform; d, dimensional; r and z, radial and vertical directions.

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